Random walks in local dynamics of network losses

I. V. Yurkevich,[1] I. V. Lerner,[1] A. S. Stepanenko,[2] and C. C. Constantinou[2]
[1]*School of Physics and Astronomy, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom*
[2]*School of Engineering, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom*
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We suggest a model for data losses in a single node (memory buffer) of a packet-switched network (like the Internet) which reduces to one-dimensional discrete random walks with unusual boundary conditions. By construction, the model has critical behavior with a sharp transition from exponentially small to finite losses with increasing data arrival rate. We show that for a finite-capacity buffer at the critical point the loss rate exhibits strong fluctuations and non-Markovian power-law correlations in time, in spite of the Markovian character of the data arrival process.

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Many systems, both natural and man-made are organized as complex networks of interconnected entities: brain cells [1], interacting molecules in living cells [2], multispecies food webs [3], social networks [4], and the Internet [5] are just a few examples. In addition to the classical Erdös-Rényi model for random networks [6], new overarching models of scale-free [7] or small-world [8] networks turn out to describe real-world examples. These and other network models have received extensive attention by physicists (see Refs. [9,10] for reviews).

A particularly interesting problem for a wide range of complex networks is their resiliency to breakdowns. The possibility of random or intentional breakdowns of the entire network has been considered in the context of scale-free networks where nodes were randomly or selectively removed [11–13], or in the context of small-world networks where a random reduction in the sites' connectivity leads to a sharp increase in the optimal distance across network which destroys its small-world nature [13–15]. In all these models, the site or bond disorder acts as an input which makes them very general and applicable to a wide variety of networks.

Network breakdowns can result not only from a physical loss of connectivity but from an operational failure of some network nodes to forward data. In the more specific class of communication networks, this could happen due to excessive loading of a single node. This could trigger cascades of failures and thus isolate large parts of the network [16]. In describing the operational failure in a particular network node, one needs to account for distinct features of the dynamically "random" data traffic which can be a reason for such a breakdown.

In this paper we model data losses in a *single node* of a packet-switched network like the Internet. There are two distinct features which must be preserved in this case: the discrete character of data propagation and the possibility of data overflow in a single node. In the packet-switched network data is divided into packets which are routed from source to destination via a set of interconnected nodes (routers). At each node packets are queued in a memory buffer before being serviced, i.e., forwarded to the next node. (There are separate buffers for incoming and outgoing packets but we neglect this for the sake of simplicity.) Due to the finite capacity of memory buffers and the stochastic nature of data traffic, any buffer can become overflown which results in packets being discarded.

We consider a model where noticeable data losses in a single memory buffer start when the average rate of random packet arrivals approaches the service rate. Under this condition the model has a built-in sharp transition from free flow to lossy behavior with a finite fraction of arriving packets being dropped. A sharp onset of network congestion is familiar to everyone using the Internet and was numerically confirmed in different models [17]. Here we stress that such a congestion can originate from a single node.

While data loss is natural and inevitable due to the data overflow, we show that loss rate statistics turn out to be highly nontrivial in the realistic case of a finite buffer, where at the critical point the magnitude of fluctuations can exceed the average value, while they obey the central limit theorem only in the (unrealistically) long-time limit. Such an importance of fluctuations in some intermediate regime is a definitive feature of mesoscopic physics, albeit the reasons for this are absolutely different (note that even in the case of electrons, the origin of the mesoscopic phenomena can be either quantum or purely classical; see, e.g., [18]). Although we model data arrivals as a Markovian process, the loss rate at intermediate times shows long-range power-law correlations in time. When excessive data losses start, it is more probable that they persist for a while, thus impacting on network operation.

The average loss rate and/or transport delays were previously studied, e.g., in theories of bulk queues [19,20] or a jamming transition in traffic flow [21]. What makes present considerations qualitatively different is that we analyze fluctuations of a discrete quantity, the number of discarded packets. Although fluctuations in network dynamics were previously studied (see, e.g., [22]), this was done in the continuous limit for the data traffic, through measurements or numerical simulations.

The mode of operation of a memory buffer is that packets arrive randomly, form a queue in the buffer, and are subsequently *serviced*. Each packet in the queue has typically a variable length and is normally serviced in fixed-length service units at discrete time intervals on a first-in first-out basis. Here we choose the simplest nontrivial model of this class: (i) packets have a fixed length of two service units; (ii) arrival and service intervals coincide. The length of the

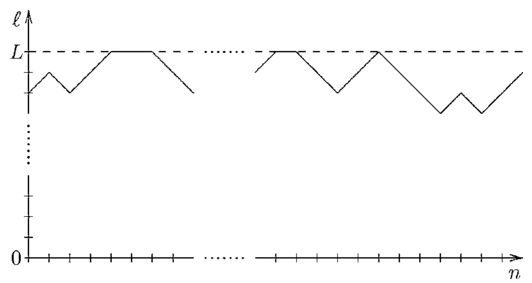

FIG. 1. The model of data losses: incoming packets randomly arrive in discrete time intervals and join the queue of length ℓ limited by the memory buffer capacity L. Packets in front of the queue are served at the same time intervals. If the queue sticks to the boundary, newly arriving packets are discarded.

queue after n service intervals, ℓ_n, serves as a dynamical variable which obeys the discrete-time Langevin equation,

$$\ell_{n+1} = \ell_n + \xi_n, \tag{1}$$

where the telegraph noise ξ_n is defined by

$$\xi_n = \begin{cases} 1, & 0 \leqslant \ell_n \leqslant L-1, \\ 0, & \ell_n = L, \end{cases} \quad \text{with probability } p$$

$$\xi_n = \begin{cases} 0, & \ell_n = 0, \\ -1, & 1 \leqslant \ell_n \leqslant L. \end{cases} \quad \text{with probability } 1-p \tag{2}$$

The above means that the length of the queue, measured in service units, either increases by one when one packet arrives and one service unit is served, or decreases by one when no packet arrives. The boundary conditions above correspond to discarding a newly arrived packet when the buffer is full ($\ell_n{=}L$) and to an idle interval when no packet arrives at an empty buffer ($\ell_n{=}0$). We show at the end of the paper that more accurate considerations, with arrival and service intervals not coinciding, do not change the asymptotic form of our results.

The main quantity which characterizes congestion is the packet loss rate which is defined via the number of packets discarded during a time interval N by

$$\mathcal{L}_N(n_0) = \sum_{n=n_0+1}^{n_0+N} \delta_{\ell_n,L}\delta_{\ell_{n+1},L}. \tag{3}$$

This means that the packet is discarded if by the moment of its arrival the queue was at the maximal capacity L as illustrated in Fig. 1. Thus the loss rate (3) is defined entirely by the processes at the boundary of the random walk (RW) so that the continuous limit cannot be exploited. This makes the loss statistics profoundly different from, e.g., the thoroughly studied statistics of first-passage time [23].

We will express the average and the variance of the loss rate (3) via the conditional probability of the queue being of length ℓ at time n provided that it was of length ℓ' at time n_0, defined by

$$\mathcal{G}_{n-n_0}(\ell,\ell') = \langle \delta_{\ell_n,\ell}\delta_{\ell_{n_0},\ell'}\rangle / \langle \delta_{\ell_{n_0},\ell'}\rangle,$$

where $\langle\ldots\rangle$ stands for the averaging over the telegraph noise of Eqs. (1) and (2). The stationary distribution of the queue length is related to $\mathcal{G}$ by

$$\mathcal{P}_{\rm st}(\ell) = \lim_{n_0\to-\infty} \mathcal{G}_{n-n_0}(\ell,\ell') = \langle \delta_{\ell_n,\ell}\rangle. \tag{4}$$

On averaging the loss rate Eq. (3), we thus obtain

$$\langle \mathcal{L}_N\rangle = \mathcal{P}_{\rm st}(L) N \mathcal{G}_1(L,L), \tag{5}$$

while its second moment is given by

$$\begin{aligned}\langle \mathcal{L}_N{}^2\rangle &= \sum_{n,m=n_0+1}^{n_0+N} \langle \delta_{\ell_n,L}\delta_{\ell_{n+1},L}\delta_{\ell_m,L}\delta_{\ell_{m+1},L}\rangle \\ &= \langle \mathcal{L}_N\rangle + 2\mathcal{P}_{\rm st}(L)\mathcal{G}_1^2(L,L)\sum_{n<m}\mathcal{G}_{m-n-1}(L,L).\end{aligned} \tag{6}$$

To calculate $\mathcal{G}$, we note that it is the Green's function of the master equation (ME) corresponding to the Langevin equation (1). The ME can be written in terms of the probability $\mathcal{P}_n(\ell)$ for the queue being of length ℓ at time n as

$$\mathcal{P}_{n+1}(\ell) = \sum_{\ell'} w_{\ell,\ell'}\mathcal{P}_n(\ell'), \quad 0 \leqslant \ell,\ell' \leqslant L. \tag{7}$$

The transition matrix $\hat{w}$ with elements $w_{\ell,\ell'}$ corresponding to Eqs. (1) and (2) is given by

$$w_{\ell,\ell'} = p\,\delta_{\ell-1,\ell'} + (1-p)\,\delta_{\ell+1,\ell'}, \quad 0 < \ell < L, \tag{8}$$

with the boundary conditions

$$w_{\ell,\ell'} = \begin{cases} (1-p)(\delta_{0,\ell'} + \delta_{1,\ell'}), & \ell = 0, \\ p(\delta_{L-1,\ell'} + \delta_{L,\ell'}), & \ell = L, \\ (1-p)\,\delta_{\ell,0} + p\,\delta_{\ell,1}, & \ell' = 0, \\ (1-p)\,\delta_{\ell,L-1} + p\,\delta_{\ell,L}, & \ell' = L. \end{cases} \tag{9}$$

Equations (7) and (8) describe the usual biased discrete-time RW on a one-dimensional lattice [23]. However, both the quantity to calculate, Eq. (3), and the boundary conditions Eq. (9) make the problem under consideration profoundly different from those in [23].

Equations (7)–(9) are clearly non-Hermitian. This leads to different right, ψ^+, and left, ψ^-, eigenfunctions of the matrix $\hat{w}$ [normalized by $\Sigma_{\ell=0}^{L}\psi^+(\ell)\psi^-(\ell)=1$]:

$$\hat{w}\psi_k^+ = \lambda_k\psi_k^+, \quad \hat{w}^T\psi_k^- = \lambda_k\psi_k^-, \tag{10}$$

where λ_k are the eigenvalues, labeled with a discrete "momentum" k. Although there exists a similarity transformation which makes the problem Hermitian (which means that all λ_k are real), it is convenient to keep the above representation unchanged.

The Green's function of the ME (7) can be expressed as $\hat{\mathcal{G}}_n = \hat{w}^n$ which gives $\mathcal{G}_n(\ell,\ell') = \Sigma_k \lambda_k^n \psi_k^+(\ell)\psi_k^-(\ell')$. Diagonalizing the tridiagonal matrix $\hat{w}$ defined by Eqs. (8) and (9), one finds the eigenvalues of Eq. (10):

$$\lambda_k = 2\sqrt{p(1-p)}\cos k, \tag{11}$$

where $k=\pi n/(L+1)$, $n=1,2,\ldots,L$. The appropriate eigenfunctions are given by

$$\psi_k^{\pm}(\ell) = c_k q^{\pm\ell/2}[\sin k(\ell+1) - q^{1/2}\sin k\ell],$$

$$c_k^2 = \frac{2p}{(L+1)(1-\lambda_k)}, \quad q \equiv \frac{p}{1-p}. \tag{12}$$

The eigenfunctions corresponding to $k=0$ are given by

$$\psi_0^{+}(\ell) = c_0 q^{\ell}, \quad \psi_0^{-}(\ell) = c_0, \quad c_0^2 = \frac{1-q}{1-q^{L+1}}, \tag{13}$$

and the appropriate eigenvalue $\lambda_0=1$ is separated by a gap from the continuous (as $L\to\infty$) spectrum of Eq. (11), unless $p=1/2$. The RW is biased toward $\ell\to L$ (full buffer and congested traffic) for $p>1/2$, or toward $\ell\to 0$ (empty buffer) for $p<1/2$. At $p=1/2$ when the RW is unbiased and the eigenvalue spectrum is gapless, the fluctuations are strongest. In all cases, since $\lambda_0=1$ while $\lambda_k<1$ for $k\neq 0$, it is the isolated solution (13) which governs the stationary distribution (4):

$$\mathcal{P}_{\rm st}(\ell) = \lim_{n_0\to-\infty} \mathcal{G}_{n-n_0}(\ell,\ell') = c_0^2 q^{\ell}. \tag{14}$$

Noticing that $\mathcal{G}_1(\ell,\ell')=w_{\ell,\ell'}$ so that $\mathcal{G}_1(L,L)=p$, one finds the average loss rate from Eqs. (5) and (14) as

$$\frac{1}{N}\langle\mathcal{L}_N\rangle = p\frac{q^{L+1}-q^L}{q^{L+1}-1} \xrightarrow[L\gg 1]{} \begin{cases} 2p-1, & p>\frac{1}{2}, \\ \frac{1}{L+1}, & p=\frac{1}{2}, \\ \frac{1-2p}{1-p}q^L, & p<\frac{1}{2}. \end{cases}$$

Thus the loss rate for $p>1/2$ is of order 1, for $p=1/2$ a small fraction of the buffer capacity, and for $p<1/2$ an exponentially vanishing function, as expected. The matching between the three asymptotic regimes takes place in a narrow region (of width $\sim 1/L$) around $p=\frac{1}{2}$.

The result for the variance Eq. (6) is conveniently expressed in terms of the "compressibility" defined by

$$\langle\delta\mathcal{L}_N^2\rangle \equiv \chi_N\langle\mathcal{L}_N\rangle, \quad \delta\mathcal{L}_N(n) = \mathcal{L}_N(n) - \langle\mathcal{L}_N\rangle. \tag{15}$$

From Eqs. (6) and (12) we find

$$\chi_N = 1 - p\mathcal{P}_{\rm st}(L) + \frac{4p(1-p)}{L+1}\sum_{k>0}\frac{\sin^2 k}{(1-\lambda_k)^2}\left(1 - \frac{1}{N}\frac{1-\lambda_k^N}{1-\lambda_k}\right). \tag{16}$$

The behavior of χ is illustrated in Fig. 2 which shows its fast increase at the critical point, $p=1/2$. Using Eq. (11) for λ_k, it is easy to simplify Eq. (16). We find that a steady-state regime (when one neglects the N-dependent term in the large parentheses above) is reached for $N\gg N_0$ where

$$N_0 \equiv [(2p-1)^2 + (\pi/L)^2]^{-1}.$$

In this regime, the compressibility saturates at

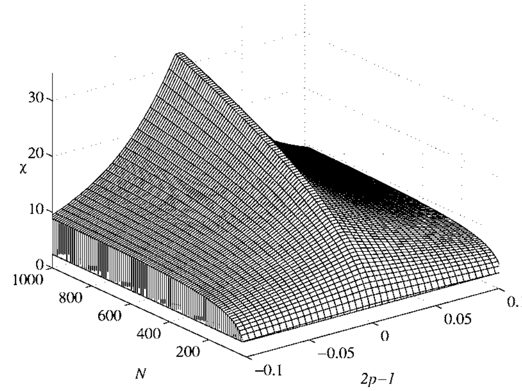

FIG. 2. Compressibility χ (for $L=1000$) shows a fast increase of fluctuations in time at the critical point $p=1/2$.

$$\chi_\infty = \begin{cases} \frac{1-|2p-1|}{|2p-1|}, & |2p-1|L \gg 1, \\ \frac{3}{2}L, & |2p-1|L \ll 1. \end{cases}$$

Although it diverges in the thermodynamic limit $L\to\infty$ and $N/L^2\to\infty$, at the transition point $p=1/2$, the variance (15) in this limit remains finite and obeys the central limit theorem.

However, this limit is reachable at the critical point only for unrealistically long times $N\gg N_0\propto L^2$. In the intermediate, practical regime $1\ll N\ll N_0$, the compressibility rapidly increases with time:

$$\chi_N = cN^{1/2}, \quad c = \frac{2\sqrt{2}}{\pi}\int_0^\infty \frac{dx}{x^2}\left(1 - \frac{1-e^{-x^2}}{x^2}\right), \tag{17}$$

so that the variance exceeds the average value of the loss rate and its distribution is no longer normal. More importantly, in this regime the fluctuations of the loss rate are no longer Markovian as they exhibit long-time correlations. To show this, we consider the temporal correlation function of the loss rate defined by

$$R_2(N,M) \equiv \frac{\langle\delta\mathcal{L}_N(0)\delta\mathcal{L}_N(M)\rangle}{\langle\delta\mathcal{L}_N^2\rangle}, \quad M>N.$$

We obtain an exact expression for $R_2(N,M)$ similar to that for χ_N, Eq. (16), omitted for brevity. In the most relevant regime $N_0\gg N\gg 1$ and $M>N$, it reduces to

$$R_2(N,M) = \frac{pN}{\chi_N}\left(e^{-M(2p-1)^2/2}\sqrt{\frac{2}{\pi M}} - |2p-1|\,\mathrm{erfc}(|2p-1|\sqrt{M/2})\right). \tag{18}$$

At the critical point this reduces using Eq. (17) to

$$R_2(N,M)|_{p=1/2} = c^{-1}\sqrt{\frac{N}{2\pi M}}. \tag{19}$$

This long-time correlation (in spite of the packet arrival being Markovian) is another clear sign of criticality.

Let us note that the boundary conditions in Eq. (9) correspond to simultaneous arrival and service of packets. In this case overflown packets are only partially discarded. In more realistic models the overflown packets should be discarded completely. To reflect this, we can choose one of the standard procedures: service first or packet arrival first. This is straightforward to formulate: the transition matrix remains the same in the bulk, Eq. (8), while it changes in 3×3 blocks in the boundary corners. In solving the eigenvalue problem (10) the appropriate boundary layer states can be eliminated. This reduces our problem to that described by Eqs. (8) and (9) but with a smaller number of states and different (and dependent on eigenvalues) corner elements on the main diagonal. This can be solved in a similar way as the model of Eqs. (7)–(9) and the dependence on N and M turns out to be the same in the asymptotical regime of Eqs. (17)–(19).

In conclusion, we have demonstrated that the stochastic nature of discrete data traffic in packet-switched networks (e.g., the Internet) results in a critical behavior with an abrupt transition from free to lossy operation at the level of a single node when the arrival rate reaches a certain critical value. The critical point is characterized by strong fluctuations and long-memory effects in the loss rate. This leads to an operational failure of a single node which can contribute to cascaded failures and thus congestion of large parts of the network. We intend to use the results of the present model as building blocks for describing such a congestion within the framework that accounts also for the topological disorder [24].

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